

THE DERIVED FUNCTORS OF THE PRIMITIVES FOR $BP_*(\Omega S^{2n+1})$

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ABSTRACT. Formulas for the Hopf invariant, and the P map in the Novikov double suspension sequence are derived. The formulas allow an effective inductive computation of the E_2 -term of the unstable Adams-Novikov spectral sequence. The 3 primary E_2 -term through the 54 stem is displayed.

1. Introduction. The unstable Adams-Novikov spectral sequence for a (simply connected) space X is a sequence of groups $E_r(X)$, $r = 2, 3, \dots$, which converge to the homotopy groups of X . It is often convenient to localize at a prime p in which case the spectral sequence converges to the homotopy groups of X localized at p , and the E_2 -term depends on the Brown-Peterson homology of X . If the (ordinary) homology of X is p -torsion free, the E_2 -term is Ext in a nonabelian category. If, in addition, the cohomology of X is a free algebra the E_2 -term may be simplified to an Ext in an abelian category [6, 7].

An important feature of this spectral sequence is the presence of EHP and double suspension long exact sequences on the E_2 level [8]. There are two single suspension sequences

$$\begin{aligned} \rightarrow E_2^s(\hat{S}^{2n}) &\xrightarrow{\sigma} E_2^s(S^{2n+1}) \xrightarrow{H'} E_2^s(S^{2pn+1}) \xrightarrow{P} E_2^{s+1}(\hat{S}^{2n}) \rightarrow, \\ \rightarrow E_2^s(S^{2n-1}) &\xrightarrow{\sigma} E_2^s(\hat{S}^{2n}) \xrightarrow{H} E_2^{s-1}(S^{2pn-1}) \xrightarrow{P} E_2^{s+1}(S^{2n-1}) \rightarrow \end{aligned}$$

and a double suspension sequence

$$\dots \rightarrow E_2^s(S^{2n-1}) \xrightarrow{\sigma^2} E_2^s(S^{2n+1}) \xrightarrow{H_2} \text{Ext}_{\mathbb{Q}_l}^{s-1}(W(n)) \xrightarrow{P_2} E_2^{s+1}(S^{2n-1}) \rightarrow \dots$$

($\hat{S}^{2n}$ and $W(n)$ are defined in [8] and §4.)

All three sequences are derived in [8] by homological methods in a nonabelian category. As a consequence it is difficult to compute the maps in the above sequences or the coaction on $W(n)$.

In this paper we study the double suspension sequence. In §4 we compute the first derived functor of the primitives of $BP_*(\Omega S^{2n+1})$ in the category of BP_* (BP)-coalgebras. This determines the coaction of $W(n)$. In §5 we use a form of the

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composite functor spectral sequence for $BP_*(\Omega S^{2n+1})$ in order to describe H_2 and P_2 . Consequently we obtain information about the single suspension sequences as there are commutative diagrams relating them to the double suspension sequence. With these results the double suspension sequence becomes an inductive method for computing the E_2 for an odd sphere. A table of the resulting computation appears in the appendix. Further applications will appear.

Throughout this paper a prime p is fixed, and $BP_*(X)$ is the (reduced) BP homology of X at the prime p . In any category under consideration I stands for the identity functor. The ring of integers is denoted by Z , the rationals by Q and the integers localized at p by $Z_{(p)}$.

2. Derived functors. We recall some of the definitions in [7]. A *cotriple* (F, δ, ϵ) on a category $\mathcal{C}$ is a functor $F: \mathcal{C} \rightarrow \mathcal{C}$ together with natural transformations $\delta: F \rightarrow F^2$, $\epsilon: F \rightarrow I$ such that the following diagrams commute.

$$\begin{array}{ccc} F & \xrightarrow{\delta} & F^2 \\ \delta \downarrow & & \downarrow \delta F \\ F^2 & \xrightarrow{F\delta} & F^3 \end{array} \quad = \quad \begin{array}{ccc} F & \xrightarrow{\delta} & F^2 \\ & \searrow & \epsilon F \downarrow \downarrow F\epsilon \\ & & F \end{array}$$

An *F-coalgebra* is an object Y of $\mathcal{C}$ together with a map $\psi: Y \rightarrow FY$ such that the following diagrams commute

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & FY \\ & \searrow & \downarrow \epsilon_Y \\ & & Y \end{array} \quad = \quad \begin{array}{ccc} Y & \xrightarrow{\psi} & FY \\ \psi \downarrow & & \downarrow F\psi \\ FY & \xrightarrow{\delta_Y} & F^2Y \end{array}$$

A map of F -coalgebras $f: (Y', \psi') \rightarrow (Y, \psi)$ is a map $f: Y' \rightarrow Y$ such that $\psi \circ f = Ff \circ \psi'$. Let $\mathcal{C}(F)$ denote the category of F -coalgebras. When the category $\mathcal{C}$ is understood we denote $\mathcal{C}(F)$ by $\mathcal{F}$. The definition of a cotriple guarantees $FY \in \mathcal{F}$ for $Y \in \mathcal{C}$. With structure map δ_Y .

A *triple* (H, μ, η) on a category $\mathcal{F}$ consists of a functor $H: \mathcal{F} \rightarrow \mathcal{F}$ together with natural transformations $\mu: H^2 \rightarrow H$, $\eta: I \rightarrow H$ such that the following diagrams commute.

$$\begin{array}{ccccc} H & \xrightarrow{H\eta} & H^2 & \xleftarrow{\eta H} & H \\ & \searrow & \downarrow \mu & \swarrow & \\ & & H & & \end{array} \quad = \quad \begin{array}{ccc} H^3 & \xrightarrow{H\mu} & H^2 \\ \mu H \downarrow & & \downarrow \mu \\ H^2 & \xrightarrow{\mu} & H \end{array}$$

Let (F, δ, ϵ) be a cotriple on $\mathcal{C}$. The *adjoint* of the cotriple (F, δ, ϵ) is a triple (H_F, μ_F, η_F) on $\mathcal{F}$ ($= \mathcal{C}(F)$) defined by $H_F Y = FY$ for $Y \in \mathcal{F}$, $\mu_F = F\epsilon$, $\eta_F = \psi_Y$.

As in [7] a cotriple F determines a functor $\mathbf{H}_F$ from $\mathcal{F}$ to the category of cosimplicial complexes over $\mathcal{F}$. Explicitly

$$\begin{aligned}\mathbf{H}_F(Y)^n &= H^{n+1}(Y), & d^i &= H^i \eta H^{n-i}: H^n \rightarrow H^{n+1}, \\ s^i &= H^i \mu H^{n-i}: H^{n+2} \rightarrow H^{n+1},\end{aligned}$$

where $(H, \mu, \eta) = (H_F, \mu_F, \eta_F)$.

$\mathbf{H}_F$ with the augmentation omitted will be denoted $\tilde{\mathbf{H}}_F$. A cosimplicial group (K^n, d_n^i, s_n^i) is *acyclic* if the homotopy of the chain complex $(K^n, \sum_i (-1)^i d_n^i)$ is zero.

Let $\mathcal{M} \subset \mathcal{F}$ be the full subcategory of objects of the form $F(C)$ with $C \in \mathcal{C}$. Let $\mathcal{A}$ be an abelian category.

Given a functor $T: \mathcal{M} \rightarrow \mathcal{A}$, $\text{ch } T\tilde{\mathbf{H}}_F(Y)$ is the cochain complex with

$$(\text{ch } T\tilde{\mathbf{H}}_F(Y))^n = T\mathbf{H}_F(Y)^n, \quad n \geq 0, \quad \text{and} \quad \delta = \sum (-1)^i T d^i.$$

If F is a cotriple on $\mathcal{C}$ the F -derived functors of T are defined by

$$R_{\mathcal{F}}^q T(Y) = H^q(\text{ch } T\tilde{\mathbf{H}}_F(Y)).$$

In the situation when the functor $\text{Hom}_{\mathcal{F}}(A, -)$ is an abelian group we use the customary notation of Ext for the derived functors of $\text{Hom}_{\mathcal{F}}(A, -)$.

3. Unstable BP . We begin by reviewing the necessary facts about BP (see [2] and [7]). Fix a prime p . There is an associative ring-spectrum BP with homology algebra $H_*(BP; Z) = Z_{(p)}[m_1, m_2, \dots]$ for canonical generators m_n , $|m_n| = 2(p^n - 1)$. Let $A = \pi_*(BP)$, $\Gamma = \pi_*(BP \wedge BP)$. Then (A, Γ) is a “Hopf algebroid”, i.e. there are structure maps consisting of a *product* $\phi: \Gamma \otimes_A \Gamma \rightarrow \Gamma$, *left*, and *right unit* maps $\eta_L, \eta_R: A \rightarrow \Gamma$, a *counit* map $\epsilon: \Gamma \rightarrow A$, and a *diagonal* $\psi: \Gamma \rightarrow \Gamma \otimes_A \Gamma$. The notation $M \otimes_A N$ requires that M be a right A -module, and that N be a left A -module. Γ is a right A -module by η_R and a left A -module by η_L [1].

The Hurewicz homomorphism $A \rightarrow H_*(BP; Z)$ is a monomorphism and a rational isomorphism. We consider m_n as being in $A \otimes Q$, and let $\log$ be the formal power series in $A[[X]] \otimes Q$ defined by

$$\log X = \sum_{i \geq 0} m_i X^{p^i}.$$

Let $\exp$ be the formal power series inverse to $\log$ characterized by $\exp \log X = X$. The formal group law for BP is given by the formal power series in $A[[X, Y]]$,

$$F(X, Y) = \sum a_{i,j} X^i Y^j = \exp(\log X + \log Y)$$

where the coefficient $a_{i,j}$ belongs to $A_{2i+2j-2}$.

Elements z_i from Γ may be substituted for the indeterminates and we write $\Sigma^F z_i$ for $F(z_1, F(z_2, \dots))$.

There is a canonical anti-isomorphism $c: \Gamma \rightarrow \Gamma$ which satisfies $c\eta_L = \eta_R$ and $c\eta_R = \eta_L$. This gives a formal group law F^* conjugate to F defined by the formula

$$\Sigma^{F^*} z_i = c(\Sigma^F c(z_i)).$$

Let $v_i \in A_{2(p^i-1)}$ be the element defined by Araki [3].

$$pm_n = \sum_{0 \leq i \leq n} m_i (v_{n-i})^{p^i}; \quad v_0 = p.$$

Let $h_i \in \Gamma_{2(p^i-1)}$ be defined by $h_i = ct_i$ where $\{t_i\}$ are the generators defined in [2]. The

$$A = Z_{(p)}[v_1, v_2, \dots], \quad \Gamma = A[h_1, h_2, \dots].$$

The right action of A on Γ is related to the left action of A on Γ by the integral form of Ravenel's Formula [13]

$$(3.1) \quad \sum F^* h_j^{p^i} \cdot v_i = \sum F^* v_j^{p^i} \cdot h_i.$$

ψ is determined by

$$(3.2) \quad \sum F^* \psi(h_i) = \sum F^* h_k^{p^j} \otimes h_j.$$

For each finite sequence of nonnegative integers $I = (i_1, i_2, \dots, i_n)$ let $h^I = h_1^{i_1} h_2^{i_2} \cdots h_n^{i_n}$. The length of I is the integer $l(I) = i_1 + i_2 + \cdots + i_n$.

DEFINITION 3.3. For each nonnegatively graded free left A -module M let $U(M)$ ($V(M)$) be the sub- A -module of $\Gamma \otimes_A M$ spanned by all elements of the form $h^I \otimes m$ where $2l(I) < \text{degree } m$ ($2l(I) \leq \text{degree } m$).

For an arbitrary left A -module M let $F_1 \xrightarrow{f} F_0 \rightarrow M \rightarrow 0$ be exact, with F_0 and F_1 free. Then define $U(M) = \text{Coker } U(f)$ ($V(M) = \text{Coker } V(f)$).

The definitions of $U(M)$ and $V(M)$ are independent of F_1 , F_0 and f .

REMARK 3.4. The natural maps from $U(M)$ or $V(M)$ to $\Gamma \otimes_A M$ are not injective if M has p -torsion. For example if M is the free $A \otimes (Z/pZ)$ module on a generator ι of dimension 3, then $ph_1^2 \otimes \iota$ is nonzero and is in the kernel of the map $U(M) \rightarrow \Gamma \otimes_A M$.

There is a Γ -comodule structure on $\Gamma \otimes_A M$ by the map $\psi \otimes 1: \Gamma \otimes_A M \rightarrow \Gamma \otimes_A \Gamma \otimes_A M$. From (3.2) it follows that $\psi \otimes 1$ takes $U(M)$ to $U^2(M)$ and $V(M)$ to $V^2(M)$, inducing maps

$$\delta^U: U(M) \rightarrow U^2(M), \quad \delta^V: V(M) \rightarrow V^2(M).$$

There are also counit maps $\epsilon^U: U(M) \rightarrow M$ and $\epsilon^V: V(M) \rightarrow M$ induced by the counit map in Γ . $(U, \delta^U, \epsilon^U)$ and $(V, \delta^V, \epsilon^V)$ are cotriples on the category of nonnegatively graded left A -modules and, by §2, define categories $\mathcal{U}$ and $\mathcal{V}$. By construction $\mathcal{U}$ and $\mathcal{V}$ are abelian categories.

Since $\Gamma_{2n+1} = 0$ there are isomorphisms

$$U(A(2n)) \simeq U(A(2n-1)) \simeq V(A(2n-1)), \quad U(A(2n+1)) \simeq V(A(2n)),$$

where $A(m)$ is a free A -module on a generator of dimension m (with trivial coaction). Hence for M a free A -module the basis of [7, §8] provides a basis of $V(M)$.

DEFINITION 3.5 $x \in V(M)$ desuspends if it is in the image of the map. $\sigma: U(M) \rightarrow V(M)$.

Let $C(M) = \text{Coker}(\sigma)$. Then $C(M)$ is free over $A \otimes Z/pZ$ if M is A -free [7, p. 245]. Some information about p torsion in $C(M)$ is provided by the following.

PROPOSITION 3.6. $h_k^n \otimes \iota_{2n}$ has order p^k in $C(A(2n))$.

PROOF. Let BP_n be the $2n$ th space in the Ω -spectrum for BP . Then by [15] $H_*(BP_n; Z_{(p)})$ is a bipolynomial Hopf algebra with indecomposables isomorphic to $V(A(2n))$ and primitives isomorphic to $U(A(2n))$. By [14] $H_*(BP_n; Z_{(p)})$ admits a decomposition

$$(*) \quad H_*(BP_n; Z_{(p)}) \simeq \bigotimes_i B_{(p)}[x_i, d_i]$$

where $B_{(p)}[x_i, d_i]$ is the universal bipolynomial Hopf algebra constructed in [10]. The lowest-dimensional term in $H_*(BP_n; Z_{(p)})$ is $\iota_{2n} \in H_{2n}(BP_n; Z_{(p)})$ (denoted b_1^{on} in [15]). Hence there is a factor $B_{(p)}[\iota, 2n]$ in $(*)$. It follows from [10, (6.1)] that there is an element of order p^k in $C(A(2n))_{2p^kn}$.

Using (3.1) we inductively show that $p^r h_r^n \otimes \iota_{2n} = 0$ in $C(A(2n))$. It suffices to show that $p^r h_r \otimes \iota_2 = 0$ in $C(A(2))$.

For $r = 1$ we have the relation $ph_1 \otimes \iota_2 = v_1 \otimes \iota_2 - 1 \otimes v_1 \iota_2$, which is zero in $C(A(2))$.

We examine the terms in dimension $2(p^r - 1)$ in relation (3.1):

$$\sum_{i,j \leq r} F^* h_j^{p^i} \cdot v_i = \sum_{i,j \leq r} F^* v_j^{p^i} h_i \quad (v_0 = p).$$

Multiplying both sides by p^{r-1} and using the inductive hypothesis we obtain

$$p^{r-1} h_r \cdot p \otimes \iota_2 = p^{r-1} \cdot p^r h_r \otimes \iota_2 \quad \text{in } C(A(2)),$$

or

$$p^r (1 - p^{p^r-1}) h_r \otimes \iota_2 = 0 \quad \text{in } C(A(2)),$$

completing the induction.

For dimension reasons it follows that the only element in dimension $2p^kn$ which can have order p^k in $C(A(2n))$ is $h_k^n \otimes \iota_{2n}$, proving (3.6). $\square$

REMARK 3.7. The indecomposable functor applied to $(*)$ gives

$$\text{Indecomposables} \left(H_*(BP_{2n}; Z_{(p)}) \right) \simeq \bigoplus_i \left(\text{Indecomposables } B_{(p)}[x_i, d_i] \right).$$

The components are invariant under the Verschiebung [15], so [10, (6.1)] determines $C(A(2n))$ completely.

Referring to the basis in [7, §8] we have the following.

COROLLARY 3.8. $\{p^{k-1} h_k^n \otimes \iota_{2n} \mid k = 1, 2, \dots\} \subset C(A(2n))$ is a set of elements independent over $A \otimes (Z/pZ)$.

PROOF. Let $U'(2n) = PH_*(BP_{2n}; Z_{(p)})$, $V'(2n) = QH_*(BP_{2n}; Z_{(p)})$, and $C'(2n) = \text{Coker}(U' \rightarrow V')$. Then from (3.6) the elements $\{p^{k-1} h_k^n \otimes \iota_{2n} \mid k = 1, 2, \dots\} \subset C'(2n)$ are nonzero elements in different degrees of $C'(2n) \otimes Z/pZ$. Filter U and V by powers of the ideal $(v_1, v_2, \dots)$, and let $E_0 U$, $E_0 V$, $E_0 C$ denote the graded groups associated to the filtration. Then $E_0 U \simeq A \otimes U'$, $E_0 V \simeq A \otimes V'$ and $E_0 C \simeq A \otimes C'$. Therefore the elements $h_1^n \otimes \iota$, $ph_2^n \otimes \iota$, $\dots$ are linearly independent over $A \otimes (Z/pZ)$ in $E_0 C$ and, therefore, also in C . $\square$

A cotriple G on the category of (-1) -connected free A -modules is constructed in [7, §6]. The associated category $\mathcal{G}$ is the category of unstable Γ -coalgebras with coalgebra structure defined in [7, (6.10)]. Finally there is the cotriple S defined on the category of positively graded free A -modules. S is the free commutative coalgebra functor. The category $\mathcal{S}$ is the same as the category of free A coalgebras (without unit). For a coalgebra with diagonal map $\Delta: M \rightarrow M \otimes_A M$ the submodule of primitives is defined by

$$P(M) = \ker(\Delta: M \rightarrow M \otimes_A M).$$

As $PG(M) \simeq U(M)$ as a coalgebra [7], the G -derived functors of P lie in $\mathcal{U}$. For $M = A(n)$, $G(M)$ is isomorphic to $SU(M)$ as coalgebras [7, (7.8)]. If $M \simeq \bigoplus A(n_i)$,

$$G(M) \simeq \bigotimes G(A(n_i)) \simeq \bigotimes SU(A(n_i)) \simeq SU\left(\bigoplus U(A(n_i))\right) \simeq SU(M)$$

where the first isomorphism follows from the definition of G [7, (6.7)]. Hence we have

$$(3.9) \quad R_{\mathcal{S}}^i PjC \simeq R_{\mathcal{G}}^i PC$$

as A -modules for $C \in \mathcal{G}$, and $j: \mathcal{G} \rightarrow \mathcal{S}$ the forgetful functor.

The derived functors of $\text{Hom}_{\mathcal{F}}(A, -)$ ($\mathcal{F} = \mathcal{G}, \mathcal{U}, \mathcal{V}$) will be abbreviated $\text{Ext}(-)$. For $W \in \mathcal{U}$, $\text{Ext}_{\mathcal{U}}(W)$ may be computed by the cobar complex $\{C^s(W), d\}$ defined in [7, §9].

The following isomorphisms relating the various Ext groups to the unstable Adams-Novikov spectral sequence are proven in [7].

$$(3.10) \quad \text{Ext}_{\mathcal{U}}(A(2n+1)) \simeq E_2(S^{2n+1})$$

and

$$\text{Ext}_{\mathcal{G}}(BP_*(X)) \simeq E_2(X)$$

when $BP_*(X)$ is A -free.

4. Derived functors of the primitives. Our objective is to compute $R_{\mathcal{G}}^i PBP_*(\Omega S^{2n+1})$ (which we will abbreviate $R_{\mathcal{G}}^i P$ in the sequel). The A -module structure has been determined in [8, (3.3)].

$$(4.1) \quad R_{\mathcal{S}}^i PBP_*(\Omega S^{2n+1}) = \begin{cases} A(2n) & \text{for } i = 0, \\ W(n) & \text{for } i = 1, \\ 0 & \text{for } i > 1, \end{cases}$$

where $A(2n)$ is the free A -module on a generator ι_{2n} of dimension $2n$, and $W(n)$ is a free $A \otimes \mathbb{Z}/p\mathbb{Z}$ module on generators $\{y_i \mid i > 0\}$ with $|y_i| = 2p^i n$.

Consider the augmented $\mathcal{G}$ -cosimplicial complex constructed in [8, §6]:

$$(4.2) \quad 0 \rightarrow BP_*(\Omega S^{2n+1}) \rightarrow G(A(2n)) \rightrightarrows G(V(A(2n))) \cdots$$

In [8, (6.1)] (4.2) is proven to be acyclic. As (4.2) is a cosimplicial resolution of $BP_*(\Omega S^{2n+1})$ by G -models, a standard double complex argument shows that (4.2) may be used to compute $R_{\mathcal{G}}^i P$. Explicitly $R_{\mathcal{G}}^i P$ is the homology of the complex.

$$\mathbf{K}: U(A(2n)) \xrightarrow{\delta_0} U(V(A(2n))) \xrightarrow{\delta_1} U(V^2(A(2n))) \rightarrow \cdots$$

where δ_* is the restriction of the differential in the stable cobar resolution of $A(2n)$ to $U(V^*(A(2n)))$.

PROPOSITION 4.3. Let $y_k \in UVA(2n)$ be defined by

$$y_k = p^{k-1}(\psi(h_k^n) - h_k^n \otimes 1) \otimes \iota_{2n}.$$

Then

$$R_g^1 P(A(2n)) \simeq (A \otimes Z/pZ)\{y_1, y_2, \dots\}.$$

PROOF. Let $\mathbf{H}$ be the complex with $\mathbf{H}^n = V^{n+1}(A(2n))$ with differentials as in §2. $\mathbf{H}$ is acyclic in dimensions > 0 being the (unaugmented) ν -cobar resolution of $A(2n)$.

The inclusion $U(A(2n)) \rightarrow V(A(2n))$ induces a short exact sequence

$$0 \rightarrow \mathbf{K} \xrightarrow{i} \mathbf{H} \xrightarrow{j} \mathbf{H}/\mathbf{K} \rightarrow 0.$$

It follows that $\partial: H^{i-1}(\mathbf{H}/\mathbf{K}) \rightarrow R_g^i P$ is an isomorphism for $i > 0$. (∂ is the boundary homomorphism of the induced long exact sequence.)

The differential $\delta_0: \mathbf{H}^0 \rightarrow \mathbf{H}^1$ is given by $\delta_0(z) = (\psi(z) - z \otimes 1) \otimes \iota_{2n}$. From the definition of ∂ it follows that

$$\partial(x) = i^{-1}(\psi(\bar{x}) - \bar{x} \otimes 1) \otimes \iota_{2n}$$

for x a cycle in $\mathbf{H}/\mathbf{K}$ and $\bar{x}$ a lift to $\mathbf{H}$.

We shall show that $\{p^{k-1}h_k^n \otimes \iota_{2n}\}$ is a set of cycles in $\mathbf{H}/\mathbf{K}$ and therefore represent elements in $H^0(\mathbf{H}/\mathbf{K})$ in dimensions $2np^k$, $k = 1, 2, \dots$. By (3.8) these elements are independent over $A \otimes (Z/pZ)$. From (4.1) $H^0(\mathbf{H}/\mathbf{K}) \cong R_g^1 P \cong W(n)$. So $\{p^{k-1}h_k^n \otimes \iota_{2n}\}$ generate $H^0(\mathbf{H}/\mathbf{K})$, and $y_k (= \partial(p^{k-1}h_k^n \otimes \iota_{2n}))$ generate $R_g^1 P(A(2n))$.

To see that $p^{k-1}h_k^n \otimes \iota_{2n} \in \mathbf{H}$ projects to a cycle in $\mathbf{H}/\mathbf{K}$ we note that $\psi(h_k) = 1 \otimes h_k + h_k \otimes 1 + \sum \gamma'_i \otimes \gamma''_i$, $\gamma'_i, \gamma''_i \in BP_*[h_1, h_2, \dots, h_{k-1}] \subset \Gamma$. As a consequence in the product

$$\psi(h_k^n) = \psi(h_k)^n = 1 \otimes h_k^n + h_k^n \otimes 1 + \sum \alpha'_i \otimes \alpha''_i$$

each term α'_i has a factor h_t with $t \leq k-1$. From (3.6) $p^{k-1}h_t$, and therefore α'_i desuspends. Hence $j\delta_0(p^{k-1}h_k^n \otimes \iota_{2n}) = 0$. $\square$

In order to compute the U structure on $R_g^1 P$ we need a more detailed description of y_k .

From (4.1) $H^1(\mathbf{K}) \cong H^1(\mathbf{K} \otimes Z/pZ)$. As observed in the proof of (4.3) the factor h_t ($t \leq k-1$) in the presence of p^{k-1} produce a desuspension of α'_i . In a similar way a factor of h_t with $t < k-1$ in the presence of p^{k-2} will produce a desuspension of α'_i . The remaining factor of p will then kill α'_i in $\mathbf{K} \otimes Z/pZ$. So the only terms in (3.2) which can contribute a nonzero term to $j^{-1}\delta_0(p^{k-1}h_k^n \otimes \iota_{2n}) \otimes Z/pZ$ are those in the formal sum

$$h_{k-1}^p \otimes h_1 + {}_{F^*}h_k \otimes 1 + {}_{F^*}h_{k-1} \otimes 1 = h_{k-1}^p \otimes h_1 + h_k \otimes 1$$

in dimension $2p^k - 2$. So we have in $\mathbf{K} \otimes Z/pZ$,

$$(4.4) \quad y_k = p^{k-1}((h_k \otimes 1 + h_{k-1}^p \otimes h_1)^n - h_k^n \otimes 1) \otimes \iota_{2n}.$$

By definition the U structure on y_k is given by applying ψ to the left factors in (4.4). So we have the following formula for the U structure on $W(n)$:

$$(4.5) \quad \psi(y_k) = p^{k-1} \left((\psi(h_k) \otimes 1 + \psi(h_{k-1})^p \otimes h_1)^n - \psi(h_k)^n \otimes 1 \right) \otimes \iota_{2n}$$

in $U(UV(A(2n)) \otimes Z/pZ)$. In order to express this in terms of $y_1, y_2, \dots$ we first compute the coefficient of y_1 .

LEMMA 4.6. $\psi(y_k) = p^{k-1} h_{k-1}^{pn} \otimes y_1 \bmod y_2, y_3, \dots$

PROOF. Let $\varepsilon: \Gamma \rightarrow A$ be the counit (see §3). The map $\varepsilon \otimes \text{id}$ restricts to a map

$$\varepsilon': UV(A(2n)) \otimes Z/pZ \rightarrow V(A(2n)) \otimes Z/pZ.$$

We have from (4.4):

$$(4.7) \quad \varepsilon'(y_k) = \begin{cases} h_1^n \otimes \iota_{2n} & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases}$$

Since $\text{Im}(\delta_0) \subset U(U(A(2n)))$, where δ_0 is the differential in $\mathbb{K}$, we must have

$$\begin{aligned} -1 \otimes y_k + \psi(y_k) \\ = \sum_{i \geq 1} \gamma_i \otimes y_i \in U(UV(A(2n)) \otimes Z/pZ) \bmod U(UU(A(2n)) \otimes Z/pZ). \end{aligned}$$

From (4.5) and (4.7) we have

$$\gamma_1 \otimes h_1^n \otimes \iota_{2n} = U(\varepsilon')(\psi(y_k)) = p^{k-1} h_{k-1}^{pn} \otimes h_1^n \otimes \iota_{2n} \bmod UU(A(2n)) \otimes Z/pZ$$

and (4.6) follows. $\square$

We can now describe $W(n)$ as an unstable Γ -comodule.

THEOREM 4.8. $\psi(y_k) = \sum p^{k-i} h_{k-i}^{np^i} \otimes y_i$.

PROOF. Let d be the differential in the stable cobar complex for $W(n)$ [7].

$$-d(y_k) = \psi(y_k) - 1 \otimes y_k.$$

By (4.6) this is equal to

$$p^{k-1} h_{k-1}^{pn} \otimes y_1 + \sum_{i=2}^{k-1} \gamma_i \otimes y_i \quad (\gamma_i \in \Gamma).$$

By (4.6) again the coefficient of y_1 in $d^2(y_k)$ is

$$\sum \gamma_i \otimes p^{i-1} h_{i-1}^{pn} - p^{k-1} d(h_{k-1}^{pn}).$$

The only terms in (3.2) which can make a nonzero contribution to $p^{k-1} d(h_{k-1}^{pn})$ are in the sum $\sum_{a+b=k-1} h_a^{p^b} \otimes h_b$. In the product

$$\begin{aligned} p^{k-1} \left(\sum_{a+b=k-1} h_a^{p^b} \otimes h_b \right)^{pn} \otimes y_1 &= p^{k-1} \sum_{a+b=k-1} h_a^{np^{b+1}} \otimes h_b^{pn} \otimes y_1 \\ &\quad + p^{k-1} (\text{mixed terms}) \otimes y_1 \end{aligned}$$

a dimension argument shows that

$$p^{k-1} (\text{mixed terms}) \otimes y_1 = 0 \bmod p.$$

As $d^2(y_k) = 0$,

$$\sum \gamma_j \otimes p^{j-1} h_{j-1}^{pn} = \sum p^{k-1} h_{k-j}^{np_j} \otimes h_{j-1}^{pn}$$

and (4.8) follows. $\square$

5. The double suspension sequence. Let D^{**} be the double complex defined as follows:

$$(5.1) \quad \begin{aligned} D^{i,j} &= U^i V^j(A(2n)), \quad i > 0, j \geq 0, \\ \partial^H: D^{i,j} &\rightarrow D^{i,j+1}, \quad \partial^V: D^{i,j} \rightarrow D^{i+1,j}, \end{aligned}$$

where ∂^H is $(-1)^i U^{i-1} \delta$ (δ is the differential in the complex $\mathbb{K}$ of §4), and ∂^V is the coboundary in the unstable cobar complex for $UV^j(A(2n))$.

By first taking vertical, then horizontal, homology we obtain $\text{Ext}_{\mathcal{V}}(A(2n))$ ($\simeq \text{Ext}_{\mathcal{U}}(A(2n+1))$) concentrated in filtration zero. By taking homologies in the opposite order we obtain a spectral sequence

$$\text{Ext}_{\mathcal{U}}(R_{\mathcal{G}}^i PBP_*(\Omega S^{2n+1})) \Rightarrow \text{Ext}_{\mathcal{U}}(A(2n+1)).$$

After identification, this is equivalent to the BP -double suspension sequence

$$(5.2) \quad \begin{aligned} \dots \xrightarrow{P_2} E_2^{s,t-1}(S^{2n-1}) &\xrightarrow{\sigma^2} E_2^{s,t+1}(S^{2n+1}) \xrightarrow{H_2} \text{Ext}_{\mathcal{U}}^{s-1,t-1}(W(n)) \\ &\xrightarrow{P_2} E_2^{s+1,t-1}(S^{2n-1}) \rightarrow \dots \end{aligned}$$

where σ^2 is the double suspension, and H_2 is the double suspension Hopf invariant. (5.2) is easily seen to be equivalent to the double suspension sequence in [8]. However (5.1) is constructed from additive functors, and does not involve the functor G . This enables us to describe the maps in (5.2).

PROPOSITION 5.3. (i) *Let $z \in \text{Ext}_{\mathcal{U}}(W(n))$ be represented in the unstable cobar complex by $\sum \gamma_k \otimes y_k$, $\gamma_k \in C^s(W(n))$. Then*

$$P_2(z) = (-1)^s \sum \gamma_k \otimes d(p^{k-1} h_k^n) \otimes \iota_{2n-1} + \sum d(\gamma_k) \otimes p^{k-1} h_k^n \otimes \iota_{2n-1},$$

where d is the differential in the stable cobar complex.

(ii) *Let $x \in E_2^s(S^{2n+1})$. Then x is represented in the unstable cobar complex by a cycle of the form*

$$\sum \gamma_k \otimes p^{k-1} h_k^n \otimes \iota_{2n+1}, \quad \gamma_k \in C^*(A(2p^k n - 1) \otimes Z/pZ)$$

modulo terms which desuspend.

$$(iii) H_2(x) = \sum \gamma_k \otimes y_k.$$

PROOF. This is a consequence of a standard diagram chase and the description of $W(n)$ in §4.

Let $\hat{S}^{2n}$ be a space with $H^*(\hat{S}^{2n}) \simeq Z[\iota_{2n}]/(\iota^p)$ [8, §1].

PROPOSITION 5.4. *There is a commutative diagram*

$$\begin{array}{ccccccc}
 \rightarrow & E_2^{s,t-1}(S^{2n-1}) & \xrightarrow{\sigma} & E_2^{s,t}(\hat{S}^{2n}) & \xrightarrow{H} & E_2^{s-1,t-1}(S^{2pn-1}) & \xrightarrow{P} & E_2^{s+1,t}(S^{2n-1}) & \rightarrow \\
 & \parallel & & \downarrow \sigma & & \downarrow r & & \parallel & \\
 \rightarrow & E_2^{s,t-1}(S^{2n-1}) & \xrightarrow{\sigma^2} & E_2^{s,t+1}(S^{2n+1}) & \xrightarrow{H_2} & \text{Ext}_{\mathfrak{q}_L}^{s-1,t}(W(n)) & \xrightarrow{P_2} & E_2^{s+1,t}(S^{2n-1}) &
 \end{array}$$

where the top row is exact, σ is the suspension, and r is the composite

$$\begin{aligned}
 E_2(S^{2pn-1}) &\simeq \text{Ext}_{\mathfrak{q}_L}(A(2pn-1)) \simeq \text{Ext}_{\mathfrak{q}_L}(A(2pn)) \\
 &\rightarrow \text{Ext}_{\mathfrak{q}_L}(A(2pn) \otimes Z/pZ) \rightarrow \text{Ext}_{\mathfrak{q}_L}(W(n)).
 \end{aligned}$$

PROOF. Let $\hat{S}^{2n} \rightarrow \Omega S^{2n+1}$ be the inclusion. By naturality there is an induced map of composite functor spectral sequences [7] and, therefore, of the equivalent long exact sequences. The only point which has to be checked is the fact that the map induces the asserted maps of derived functors. This follows from the map induced on the injective extension sequences used to compute the derived functors. See [8] for details. $\square$

COROLLARY 5.5. (i) $P: E_2(S^{2pn-1}) \rightarrow E_2(S^{2n-1})$ is given by

$$P(\gamma \otimes \iota) = \pm \gamma \otimes d(h_1^n) \otimes \iota_{2n-1},$$

i.e. P is composition with the (mod p) Whitehead product.

(ii) Let $x \in E_2(S^{2n+1})$. x desuspends to $E_2(\hat{S}^{2n})$ if and only if x can be represented in the unstable cobar complex by $\gamma \otimes h_1^n \otimes \iota_{2n+1}$ ($\gamma \in C^*(A(2pn-1))$) modulo terms which desuspend to $E_1(S^{2n-1})$.

(iii) Let $x' \in E_2(\hat{S}^{2n})$ be a desuspension of x . Then, in the notation of (ii),

$$H(x') = \gamma \otimes \iota_{2pn-1} \pmod{p}.$$

It follows that P commutes with the differentials in the unstable Adams-Novikov spectral sequence (see [6, (4.9)]).

REMARK 5.6. To see that H commutes with the differentials in the unstable Adams-Novikov spectral sequence we consider the spectral sequence for $\Omega \hat{S}^{2n}$.

There is an isomorphism

$$E_2(\Omega \hat{S}^{2n}) \simeq E_2(S^{2n-1}) \oplus E_2(\Omega S^{2pn-1})$$

with a nontrivial d_2 ,

$$d_2: E_2^{s,t}(\Omega S^{2pn-1}) \rightarrow E_2^{s+2,t+1}(S^{2n-1}),$$

determined by

$$d_2(x_{2pn-2}) = d(h_1^n) \otimes \iota_{2n-1}$$

where x_{2pn-2} generates $BP_{2pn-2}(\Omega S^{2pn-1})$, ι_{2n-1} generates $BP_{2n-1}(S^{2n-1})$ and

$$d(h_1^n) = \sum \binom{n}{i} h_1^i \otimes h_1^{n-i}.$$

d_2 is the “ P -map” in the EHP sequence induced in E_3 .

$$\begin{array}{ccccccc}
\begin{array}{c} d_2 \\ \rightarrow \end{array} & E_2(S^{2n-1}) & \rightarrow & E_3(\Omega \hat{S}^{2n}) & \xrightarrow{h} & E_2(\Omega S^{2pn-1}) & \begin{array}{c} d_2 \\ \rightarrow \end{array} \\
& \parallel & \nearrow & & \searrow & \parallel & \\
& E_3(S^{2n-1}) & & & & E_3(S^{2pn-1}) &
\end{array}$$

where the top row is the consequence of the definition of E_3 as the homology of (E_2, d_2) , and h is induced by a map of spaces and therefore commutes with differentials.

For a space X the natural map $BP(\Omega X) \rightarrow \Omega BP(X)$ induces a map of spectral sequences

$$\omega_X: E_r^{*,*-1}(\Omega X) \rightarrow E_r^{*,*}(X).$$

By [8, (6.1)] $\omega_{S^{2n-1}}$ is an isomorphism.

For $X = \hat{S}^{2n}$ $\omega_{\hat{S}^{2n}}$ induces a nonfiltration preserving map ω . ω is determined by

$$\omega(\iota_{2n-1}) = \iota_{2n}, \quad \omega(x_{2pn-2}) = h_1^n \otimes \iota_{2n}$$

where ι_{2n} generates $BP_{2n}(\hat{S}^{2n})$.

ω occurs in the middle column of the following commutative diagram

$$\begin{array}{ccccccc}
E_2(S^{2n-1}) & \xrightarrow{\sigma} & E_2(\hat{S}^{2n}) & \xrightarrow{H} & E_2(S^{2pn-1}) & \xrightarrow{P} & \dots \\
& & \parallel & & \parallel & & \\
& & E_3(\hat{S}^{2n}) & & E_3(S^{2pn-1}) & & \\
& \parallel & \uparrow & & \uparrow \parallel & & \\
E_3(S^{2n-1}) & \rightarrow & E_3(\Omega \hat{S}^{2n}) & \xrightarrow{h} & E_3(\Omega S^{2pn-1}) & \xrightarrow{d_2} &
\end{array}$$

It follows from the 5-lemma that ω is an isomorphism of spectral sequences (for $r \geq 3$), and H therefore commutes with unstable Adams-Novikov differentials. $\square$

REMARK 5.7. (i) $E_2^{s,t}(S^{2n+1})$ is mod p meta-stable if

$$t < \begin{cases} 2(p-1)pk + 2p^2n + 2, & s = 2k + 1, \\ 2(p-1)(pk + 1) + 2p^2n + 2, & s = 2k + 2. \end{cases}$$

In the mod p meta-stable range there is a commutative ladder of exact sequences

$$\begin{array}{ccccccc}
\rightarrow & E_2(\hat{S}^{2n}) & \xrightarrow{\sigma} & E_2(S^{2n+1}) & \xrightarrow{H'} & E_2(S^{2pn+1}) & \xrightarrow{P} \dots \\
& \downarrow H & & \downarrow H_2 & & \parallel \sigma^2 & \\
\rightarrow & E_2(S^{2pn-1}) & \xrightarrow{r} & \text{Ext}_{\mathbb{Q}_l}(A(2pn-1) \otimes Z/pZ) & \xrightarrow{\delta} & E_2(S^{2pn-1}) & \xrightarrow{xp} \dots
\end{array}$$

where the bottom row is the Bockstein sequence, and H' is induced by the James map [8]. H' is therefore computed by (5.3) and the Bockstein differential (in the meta-stable range).

(ii) There is a spectral sequence

$$\bigoplus_{i \geq 1} \text{Ext}_{\mathbb{Q}_l}(A(2p^i n - 1) \otimes Z/pZ) \Rightarrow \text{Ext}_{\mathbb{Q}_l}(W(n))$$

with differentials given by (4.8). Together with the universal coefficient theorem and the results of §5 we have an inductive method for computing $E_2(S^{2n+1})$ which is similar to the computation in [16].

(iii) (5.3) is not useful for some infinite computations. For example, in order to compute the Hopf invariant of the stable elements constructed in [12] it is convenient to introduce an unstable version of the chromatic filtration (see [4]).

Appendix. With the formulas of §§4 and 5 the unstable Adams-Novikov spectral sequence becomes a powerful tool for computing the homotopy groups of odd spheres (at least for $p > 2$). Following is a table of the 3-primary unstable Adams-Novikov E_2 term through the 54-stem. (The classical unstable Adams spectral sequence has approximately 3 times as many elements.)

Notation. (i) Elements in the table are listed by leading term in the sphere of origin filtration.

(ii) $(xh_1^n)_{yh_1^m}$ denotes an element on the $2n + 1$ sphere which is killed or made homologous to another class on the $2m + 1$ sphere. If the condition of (5.3)(ii) is satisfied x and y are the Hopf invariants (see Remark (i) below).

(iii)

$$\begin{array}{c} X \\ \vdots \\ Y \end{array}$$

denotes an extension.

(iv) For x a cycle, $\bar{x}$ denotes a cochain with $d\bar{x} = 3x$.

(v) In filtration 2, $\{\alpha_n h_1^m\}_{h_1^{n+m}}$ denotes the tower

$$\begin{array}{c} (\tilde{\alpha}_{n+m-1} h_1)_{v_1^{m-1} h_1^{n+1}} \\ \vdots \\ (\tilde{\alpha}_{n+1} h_1^{m-1})_{v_1 h_1^{n+m-1}} \\ \vdots \\ (\tilde{\alpha}_n h_1^m)_{h_1^{n+m}} \end{array}$$

where $\tilde{\alpha}_k$ is the generator of the image of the J homomorphism in the $2(p-1)k-1$ stem in filtration 1.

(vi) We adopt the notation of [12] for the stable elements in filtrations 1 and 2. Their Hopf invariants are given by

$$\begin{aligned} \alpha_{m/n} &= v_1^{n-m} h_1^n \quad (\alpha_{m/1} = \alpha_m), & \beta_1 &= \alpha_1 h_1^2, \\ \beta_i &= \overline{\beta_{i-1}} h_1^4, \quad i > 1, & \beta_{3/3} &= \alpha_2 h_1^7, & \beta_{3/2} &= \bar{\beta}_1 h_1^7. \end{aligned}$$

(vii) The class $\alpha_2 h_1^4 \circ h_1^2$ in the 29 stem survives to $\beta_2 \alpha_1$.

The class $\beta_2 h_1^3$ suspends to $\beta_{3/3} \alpha_1 (= \epsilon')$ in the 37-stem.

The class $\beta_{3/3} \circ h_1^3$ in the 45-stem survives to the element ϕ in homotopy.

In the 48-stem the class $\beta_{3/3} \alpha_1 h_1^3$ does not suspend to zero in homotopy from S^9 .

In fact the class jumps filtration and suspends to $\beta_1^3 \alpha_1 h_1^4$.

In a similar way the class $\beta_{3/3}v_1h_1^3$ in the 49-stem suspends to $\beta_1^2\alpha_1\beta_2$. The class $\beta_{3/3}v_1h_1^4$ in the 53-stem suspends to $\beta_1^3v_1h_1^5$.

REMARKS. (i) For elements in the tower defined in (v) above, one must be careful when reading the Hopf invariants of elements not in the meta-stable range (see 5.7). For example $\alpha_{9/3}h_1 \in E_2^{2,43}(S^3)$ is not in the form required by 5.3(ii) ($\alpha_{9/3} \notin C^*(A(5) \otimes Z/pZ)$). We use the relation $\alpha_{9/3}h_1 = \alpha_8\alpha_2$ to compute $H_2(\alpha_{9/3}h_1) = \alpha_8v_1$. This is the first example of an unstable element in filtration two which is born on an odd sphere. A complete description of the unstable elements in filtration two will appear elsewhere.

(ii) If $\gamma = \sigma^2\delta$ then $H_2(\gamma x) = \gamma H_2(x)$ (see 5.3(ii)).

Low-dimensional computations. For convenience we desuspend the generators y_i of $W(n)$ and use the isomorphism $\text{Ext}_{\mathbb{Q}_l}(W(n)) \simeq \text{Ext}_{\mathbb{Q}_l}(\sigma^{-1}W(n))$. We denote $\sigma^{-1}y_i$ by $x_{2p'n-1}$ and, by abuse of notation, denote $\sigma^{-1}W(n)$ by $\{x_{2pn-1}, x_{2p^2n-1}, \dots\}$ where the only generators to be indicated will be those which are relevant for the range of computation being considered. For stem $t - s$ we compute the following part of the double suspension sequence:

$$(6.1) \quad \text{Ext}^{s-2, t+2n-1}(\{x_{2pn-1}, \dots\}) \xrightarrow{P} E_2^{s, t+2n-1}(S^{2n-1}).$$

Both terms are inductively known. Elements in the image of P suspend to zero on S^{2n+1} . Elements in the kernel of P produce elements on S^{2n+1} in the $s - t + 1$ stem. The stem of $\text{Ext}_{\mathbb{Q}_l}^{s,t}(\sigma^{-1}W(n))$ is defined to be $t - s - (2pn - 1)$. As n increases to $n + 1$ in (6.1) the stem of $\text{Ext}_{\mathbb{Q}_l}(\sigma^{-1}W(n))$ decreases by $2(p - 1)$. P increases the stem by $2n(p - 1) - 2$.

Finally we note that $E_2^{s,t}(S^1) = 0$ if $(s, t) \neq (0, 1)$.

The first stem greater than zero where there can be a nonzero group in (6.1) is the 3-stem with $n = 1, s = 2$.

There is an element produced in the 3-stem on S^3 . The leading term of the generator is given taking the inverse of the map H_2 . In this case we produce the generator h_1 , which we denote by α_1 .

For dimension reasons there are no other elements in this stem.

The next nonzero groups occur in the 6-stem. For $n = 1$ the P map increases stem by 2. By induction the domain of the P -map is generated by α_1 , and we have an element α_1^2 in the 6-stem on S^3 .

In stem 6 the domain of the P -map for $n = 1$ is the 4-stem which is generated in filtration zero by $v_1 (= \bar{\alpha}_1)$. This produces an element with leading term v_1h_1 in the 7-stem which we call α_2 .

For $n = 2$ the P -map shifts dimension by 6, $d(x_{11}) = \alpha_1^2$ (up to a unit). So α_1^2 dies on S^5 .

The next interesting stem is the 10-stem. For $n = 1$ $P(\alpha_2) = 0$, producing $\alpha_2\alpha_1$ on S^3 in the 10-stem.

The stem shift for $n = 2$ is 6. So we need to compute $P(\alpha_1x_{11}) = \alpha_1^3 = 0$. Therefore an element β_1 with leading term $\alpha_1h_1^2$ is produced in the 10-stem on S^5 . Furthermore $3 \cdot \alpha_1h_1^2 = h_1 \cdot v_1h_1 = v_1h_1 \cdot h_1 = \alpha_2\alpha_1$ modulo terms which desuspend, hence we obtain the extension indicated in the table.

The method of computation described thus far is sufficient to complete the table through the 12-stem. In the 13-stem there is the possibility of the first differential δ_1 in the spectral sequence (5.7(ii)).

For $n = 1$ we have

$$\delta_1: \text{Ext}^{0,17}(A(17) \otimes Z/pZ) \rightarrow \text{Ext}^{1,17}(A(5) \otimes Z/pZ) \simeq Z/pZ\{\alpha_{3/2}, \overline{\alpha_2\alpha_1}\}$$

(the computation of $\text{Ext}^{1,17}(A(5) \otimes Z/pZ)$ is inductively determined from the table).

With d the differential in the unstable cobar complex of $W(1)$ we have from (4.5),

$$\delta_1(x_{17}) = ph_1^3 \otimes x_5 = \overline{\alpha_2\alpha_1} \otimes x_5$$

(to see this we compute, using the table, $d(ph_1^3) = p(\alpha_2\alpha_1)$). We now have $\text{Ext}^{1,17}(W(1)) \simeq Z/pZ\{\alpha_{3/2}\}$, and an element with Hopf invariant $\alpha_{3/2}$ is born on the 3-sphere in the 14-stem. (It is convenient to observe that the action (4.8) is unstable, so $\delta_1(x_{17})$ cannot be $\alpha_{3/2}$.)

Computations support the following conjecture (also observed by H. Miller):

CONJECTURE 6.2. The spectral sequence (5.7(ii)) collapses at E_2 , and the natural map $\text{Ext}_{\mathbb{Q}_l}(A(2pn-1) \otimes Z/pZ) \rightarrow \text{Ext}_{\mathbb{Q}_l}(W(n))$ is surjective.

6.2 is a useful “working principle” which can help motivate differentials in 5.7. Partial results relating to 6.2 will appear elsewhere.

It is convenient to use the stable Novikov spectral sequence [11] to simplify the computations on large stems. For example, in the 52-stem, $(\beta_2)^2$ generates a stable Z/pZ . In order to get the correct result $P(\beta_2 v_1)$ must be ϕh_1^2 or $\phi h_1^2 - \beta_2^2$ on S^{13} . A stable computation mod p shows that it is the latter.

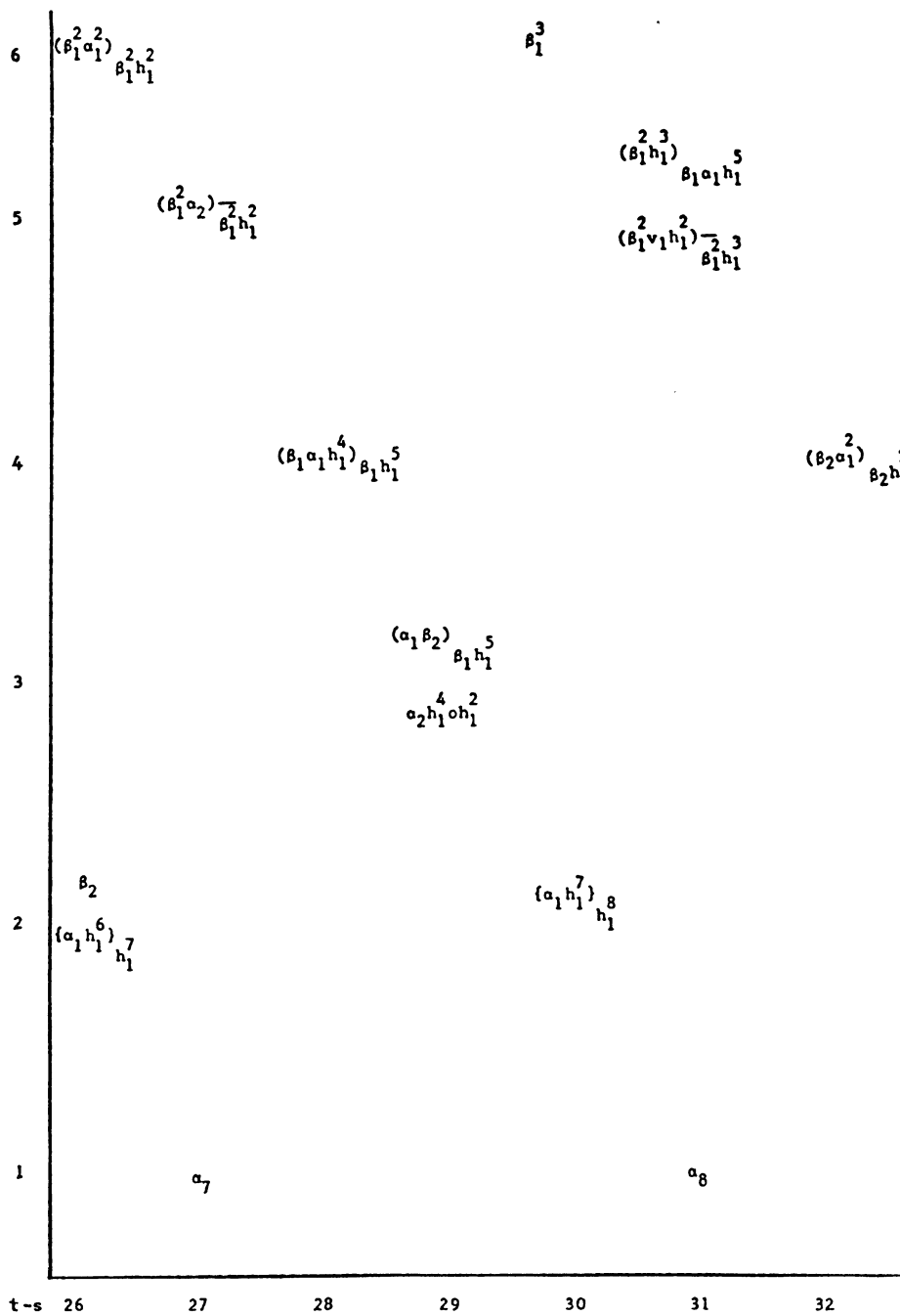
DIFFERENTIALS. Differentials are determined by pulling back stable differentials, the multiplicative properties of the spectral sequence [6, (4.9)] or by 5.6.

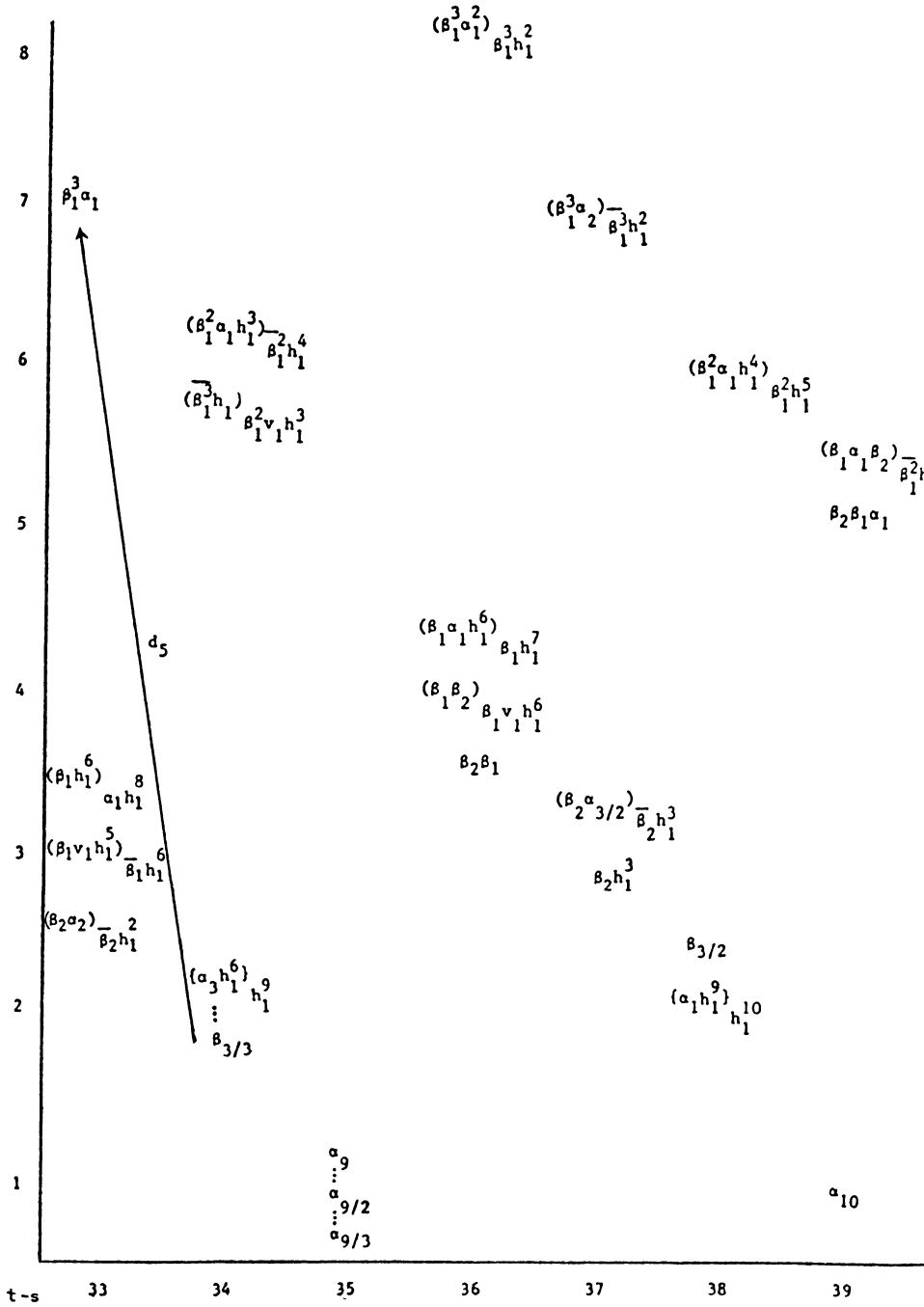
As a consequence of 5.6 the d_2 in the spectral sequence for $\Omega\hat{S}^{2n}$ satisfies $d_2(x) = P(H(x))$ for $x \in E_2(\Omega\hat{S}^{2n})$. We conjecture the same remark is true for the unstable higher differentials in the spectral sequence for $\Omega\hat{S}^{2n}$, and by 5.6 for the unstable higher differentials in the spectral sequence for S^k .

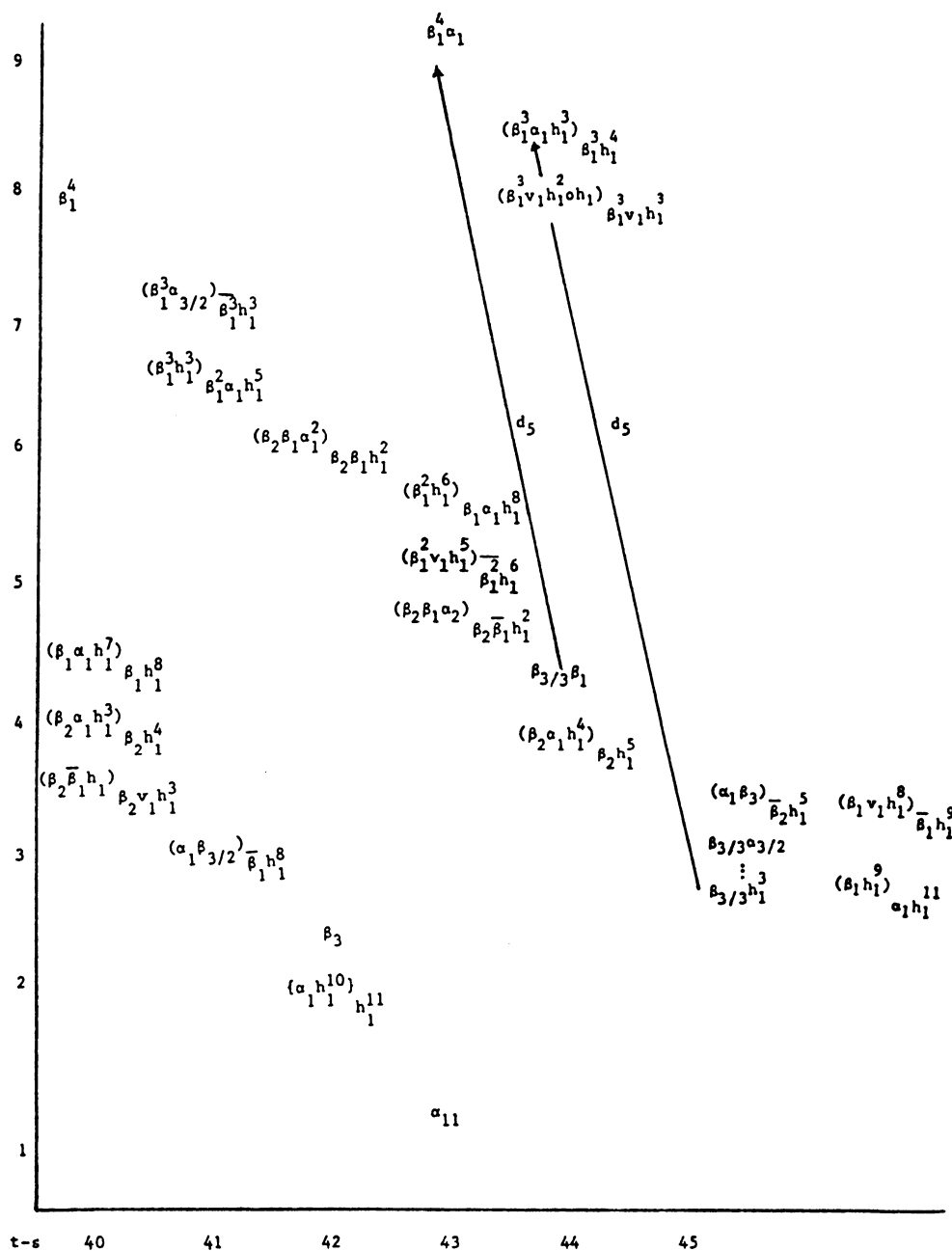
For example the element $\phi (= \beta_{3/3} h_1^3)$ in the 45-stem is born in homotopy on S^9 . The target of the differential is $P(\beta_1^3)$. The conjecture would imply that $H(\phi) = \beta_1^3$ in homotopy.

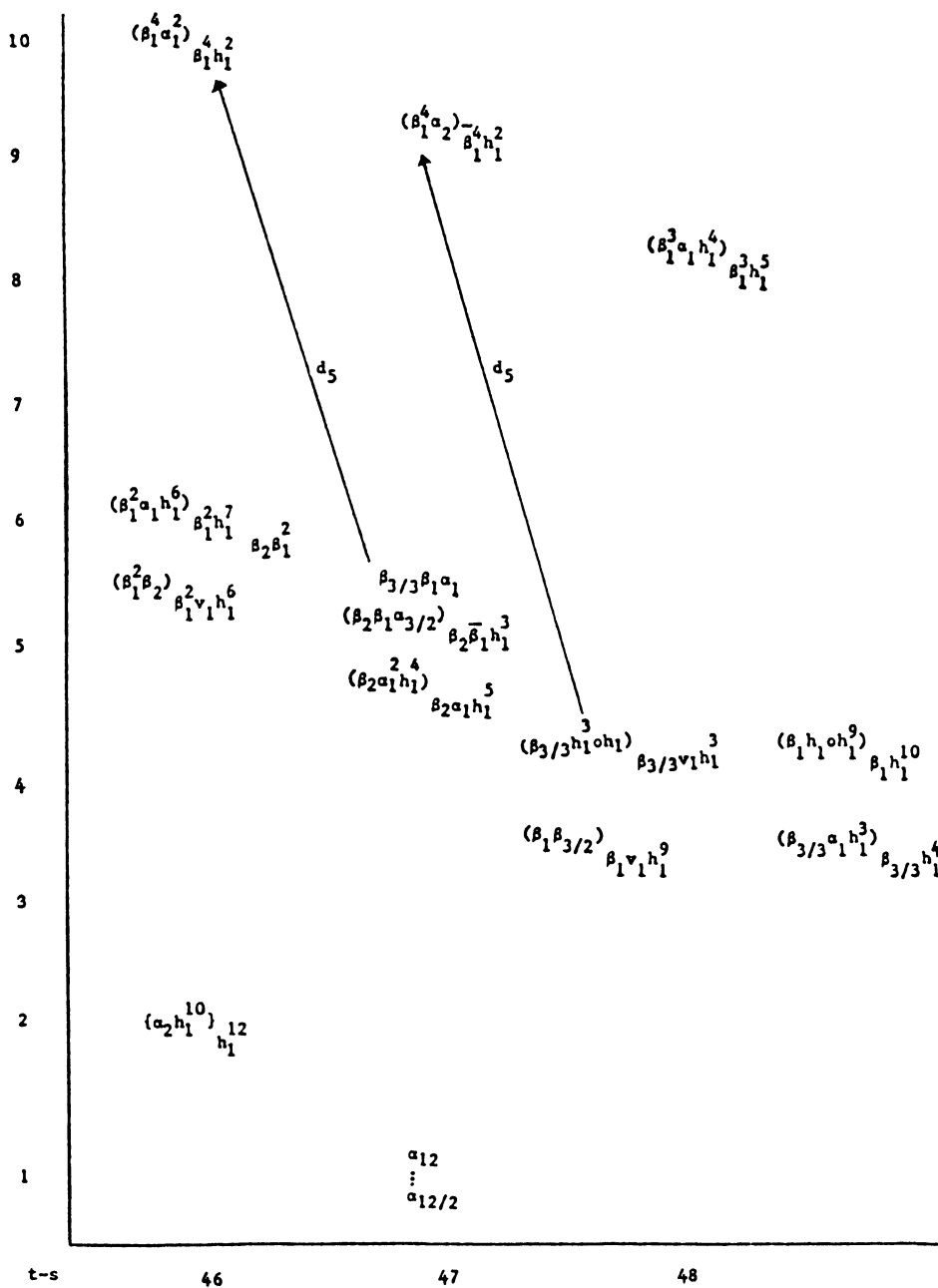
This may be seen from the table. The Hopf invariant is in the 30-stem in filtration 3 or more. The only possibility is β_1^3 .

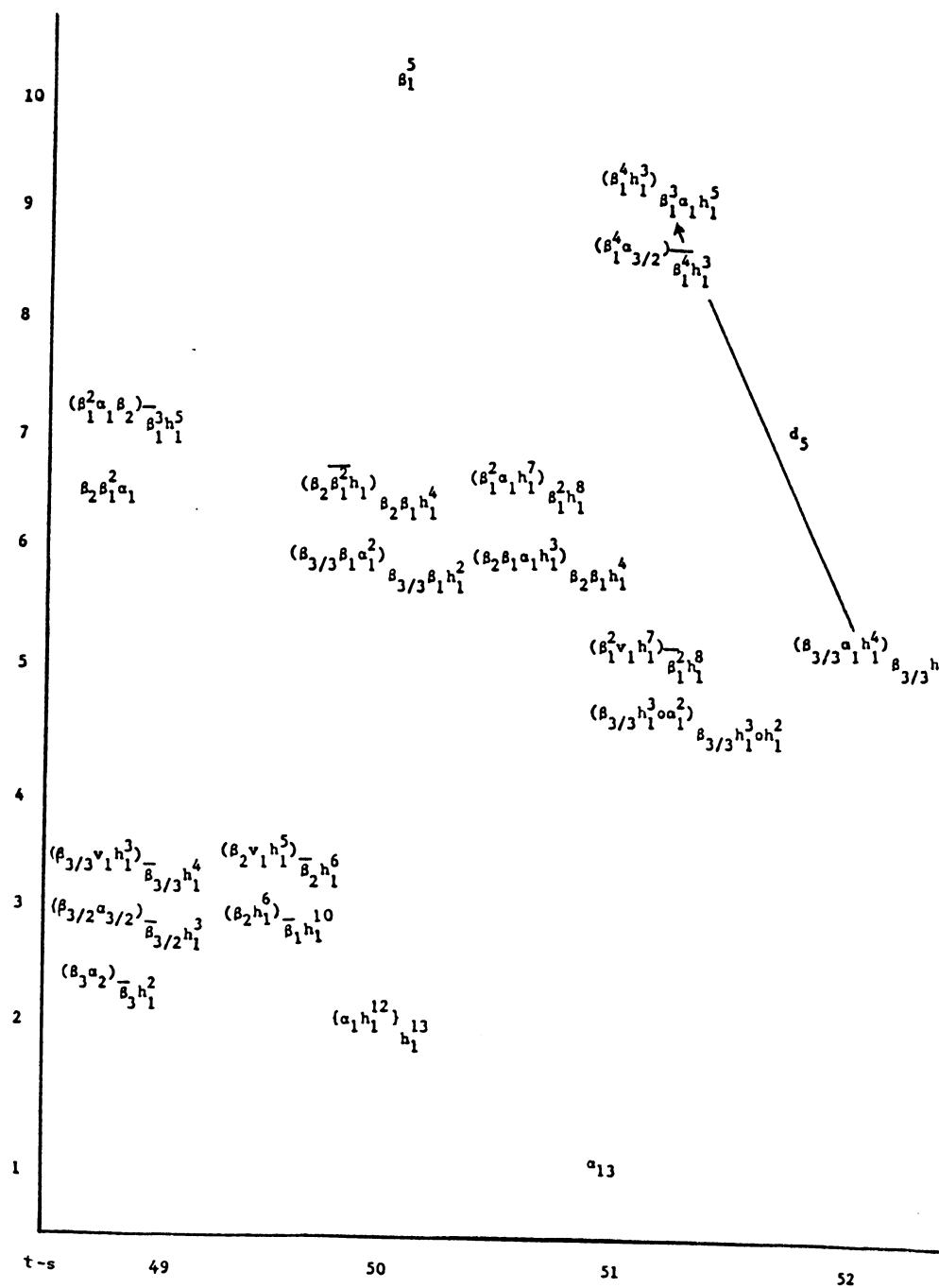
If $d_5(\beta_{3/3})$ were zero, and not $\beta_1^3\alpha_1$, then the double suspension sequences in homotopy and in Ext would be the same through the range of the table. A computation through the 52-stem which assumed $\beta_{3/3}$ is a stable homotopy element was made by Brayton Gray [9].











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